On the true Value of the integral formula $\int \partial x \left(\log \frac{1}{x}\right)^n$ extended from the lower limit $x = 0$ to the upper limit $x = 1$ *

Leonhard Euler

§1 Since this formula expresses the area under a transcendental curve, to whose abscissa x the ordinate $= \left(\log \frac{1}{x}\right)^n$ corresponds, the question reduces to this that the same area, provided it corresponds to the abscissa $x = 1$, is exhibited either by absolute numbers or at least by means of quadratures of algebraic curves. And at first it is certainly obvious, as often as n was an integer, that this integral formula yields the general term of the hypergeometric progression, since it is

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$$\int \partial x \left(\log \frac{1}{x}\right)^0 = 1,$$
$$\int \partial x \left(\log \frac{1}{x}\right)^1 = 1,$$
$$\int \partial x \left(\log \frac{1}{x}\right)^2 = 1 \cdot 2,$$
$$\int \partial x \left(\log \frac{1}{x}\right)^3 = 1 \cdot 2 \cdot 3,$$
$$\int \partial x \left(\log \frac{1}{x}\right)^4 = 1 \cdot 2 \cdot 3 \cdot 4$$

and hence in general

$$\int \partial x \left(\log \frac{1}{x}\right)^n = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n,$$

which value is indeed only seen, if n was a positive integer number; furthermore, if n was a negative integer number, from the nature of the hypergeometric series it is on the other hand easily understood that all values of our formula become infinitely large. Therefore, the question here mainly contains the cases, in which the number n is a fractional number; for, in these cases it is certainly not possible to assign the value of our formula in terms of absolute numbers by any means, but rather quadratures of algebraic curves of the higher orders the greater the denominator of the fraction n was assumed are required, as I demonstrated once already[1]. But recently I discovered a new method to investigate the same transcendental values, which method I decided to explain here, since hence many results useful for the whole field of Analysis follow from this.

§2 Therefore, first of all for the sake of brevity let us put

$$\log \frac{1}{x} = u,$$

[1]Euler refers to his paper "Evolutio formulae integralis $\int x^{f-1} dx \left(\log x\right)^{\frac{n}{m}}$ integratione a valore $x = 0$ ad $x = 1$ extensa". This is paper E421 in the Eneström-Index.

that it is $\partial u = -\frac{\partial x}{x}$ and hence $\partial x = -x\partial u$. Hence one can immediately derive extraordinary reductions by means of the well known lemma which states that it is

$$\int P\partial Q = PQ - \int Q\partial P.$$

For, having taken

$$P = u^n \quad \text{and} \quad \partial Q = \partial x$$

because of

$$\partial P = nu^{n-1}\partial u = -\frac{nu^{n-1}\partial x}{x} \quad \text{and} \quad Q = x$$

this lemma gives us

$$\int u^n \partial x = xu^n + n\int u^{n-1}\partial x.$$

Further, because it is $u^n\partial x = -xu^n\partial u$, if one here puts

$$P = -x \quad \text{and} \quad \partial Q = u^n\partial u,$$

because of

$$\partial P = -\partial x \quad \text{and} \quad Q = \frac{1}{n+1}u^{n+1}$$

we will have

$$\int u^n\partial x = -\frac{1}{n+1}xu^{n+1} + \frac{1}{n+1}\int u^{n+1}\partial x.$$

Hence, since these integrals must be taken in such a way that they vanish having put $x = 0$, but then one has to put $x = 1$, it is known that the absolute terms in this reduction become zero such that for this case, which the only one considered here, it is

$$\int u^n\partial x = n\int u^{n-1}\partial x,$$

but then also

$$\int u^n\partial x = \frac{1}{n+1}\int u^{n+1}\partial x,$$

which last reduction follows directly from the first, of course.

§3 But that the formula xu^n always vanishes in the case $x = 0$, is usually not demonstrated directly and could even be in doubt, since having put $x = 0$ it is $u^n = \infty$; but on the other hand this statement can be proved rigorously in the following way. For, for the case $x = 0$ let us set

$$xu^n = v,$$

so that we have to explore the value of this letter v, which we therefore want to represent this way by means of a fraction

$$v = \frac{x}{u^{-n}},$$

whose numerator and denominator vanish in the case $x = 0$, whence by the general rule so instead of the numerator as instead of the denominator we have to write their differentials, and since the same value of v has to result, it will also be

$$v = \frac{\partial x}{-nu^{-n-1}\partial u} = \frac{+x}{nu^{-n-1}} \quad \left(\text{because of} \quad \partial u = -\frac{\partial x}{x}\right).$$

Therefore, because from the first value it is $v = xu^n$, but from the second $v = \frac{1}{n}xu^{n+1}$, it will hence be

$$v^{n+1} = x^{n+1}u^{n(n+1)},$$

and from the other

$$v^n = \left(\frac{x}{n}\right)u^{n(n+1)};$$

the second of these values divided by the first will give $v = n^n x$, and this expression must equally exhibit the true value of v for the case $x = 0$, but having put $x = 0$ it obviously is $v = 0$.

§4 Since our investigation here is mainly restricted to the cases, in which the exponent n is a fraction, by means of the reduction $\int u^n \partial x = n \int u^{n-1}\partial x$ all fractions assumed for n, no matter how large they were, can continuously decreased by 1 and hence finally can be made smaller than 1 so that n is contained within the limits 0 and 1. Further, by means of the other reduction $\int u^n \partial x = \frac{1}{n+1}\int u^{n+1}\partial x$, if the exponent n was a negative fraction, its value can finally equally be reduced to a fraction within the limits 0 and 1; hence here it will be sufficient for us to have expanded only the cases, in which the fractions assumed for n lie within the limits 0 and 1; and these fractions are conveniently subdivided into various classes, depending on whether the denominators of these fractions were either 2 or 3 or 4 or 5 etc.

§5 After I had recently contemplated series formed from binomial coefficients[2], I showed, if one puts

$$(1+z)^m = 1 + Az + Bz^2 + Cz^3 + \text{etc.},$$

but then also

$$(1+z)^n = 1 + \alpha z + \beta z^2 + \gamma z^3 + \text{etc.},$$

that then the sum of this series

$$1 + A\alpha + B\beta + C\gamma + \text{etc.} = s$$

can be expressed in such a way that it is

$$s = \frac{\int u^{m+n}\partial x}{\int u^m \partial x \cdot \int u^n \partial x},$$

which sum is therefore defined by means of the propounded integral formula; further, I also showed that the same sum can also be expressed this way

$$s = \frac{m+n}{mn\int x^{m-1}\partial x(1-x)^{n-1}},$$

whence it therefore follows that it will always be

$$\frac{m+n}{mn}\int u^m\partial x \cdot \int u^n \partial x = \int u^{m+n}\partial x \cdot \int x^{m-1}\partial x(1-x)^{n-1},$$

if these single integrals are extended from the lower limit $x=0$ to the upper limit $x=1$, of course.

[2]Euler refers to his paper "Plenior expositio serierum illarum memoragilium, quae ex unciis potestatum binomii formantur". This is paper E663 in the Eneström-Index.

§6 But since the present undertaking is about fractions and they are smaller than 1, let us in general put $m = \frac{\mu}{\lambda}$ and $n = \frac{\nu}{\lambda}$, so that it is

$$\frac{\lambda(\mu+\nu)}{\mu\nu}\int u^{\frac{\mu}{\lambda}}\partial x \cdot \int u^{\frac{\nu}{\lambda}}\partial x = \int u^{\frac{\mu+\nu}{\lambda}}\partial x \cdot \int x^{\frac{\mu-\lambda}{\lambda}}\partial x(1-x)^{\frac{\nu-\lambda}{\lambda}}.$$

But now in order to get rid of the fractional exponents in the last integral formula, let us set $x = z^{\lambda}$ and because of $\partial x = \lambda z^{\lambda-1}\partial z$ it will be

$$\int x^{\frac{\mu-\lambda}{\lambda}}\partial x(1-x)^{\frac{\nu-\lambda}{\lambda}} = \lambda\int z^{\mu-1}\partial z(1-z^{\lambda})^{\frac{\nu-\lambda}{\lambda}},$$

which formula because of $\nu - \lambda < 0$ can be represented this way

$$\lambda\int\frac{z^{\mu-1}\partial z}{\sqrt[\lambda]{(1-z^{\lambda})^{\lambda-\nu}}},$$

which integral equally is to be extended from $z = 0$ to $z = 1$. Therefore, after this substitution our principal equation will be

$$\frac{\mu+\nu}{\mu\nu}\int\partial x\sqrt[\lambda]{u^{\mu}}\cdot\partial x\sqrt[\lambda]{u^{\nu}} = \int\partial x\sqrt[\lambda]{u^{\mu+\nu}}\cdot\int\frac{z^{\mu-1}\partial z}{\sqrt[\lambda]{(1-z^{\lambda})^{\lambda-\nu}}},$$

where the two numbers μ and ν will always be positive and smaller than λ for us. But here especially the fact deserves it to be mentioned that in the case, in which it is $\mu + \nu = \lambda$, the last integral can always be reduced to the quadrature of the circle in such a way that it is

$$\int\frac{z^{\mu-1}\partial z}{\sqrt[\lambda]{(1-z^{\lambda})^{\lambda-\nu}}} = \frac{\pi}{\lambda\sin\frac{\mu\pi}{\lambda}}.$$

§7 Now from this principal equation without any difficulty one will find the values of the propounded integral formula for the single denominators λ, if only in each case successively all numbers smaller than the denominator λ are attributed to the letters μ and ν; for, then many equations will be formed and using them one will be able to define the values of the formulas $\int\partial x\sqrt[\lambda]{u^{\mu}}$ and $\int\partial x\sqrt[\lambda]{u^{\mu}}$. But concerning the formula $\int\partial x\sqrt[\lambda]{u^{\mu+\nu}}$, which resulted from $\int u^{m+n}\partial x$, whenever it was $m + n > 1$ or $\mu + \nu > \lambda$, since we saw that it is $\int u^{m+n}\partial x = (m+n)\int u^{m+n-1}\partial x$, it will be

$$\int\partial x\sqrt[\lambda]{u^{\mu+\nu}} = \frac{\mu+\nu}{\lambda}\int\partial x\sqrt[\lambda]{u^{\mu+\nu-\lambda}},$$

which formula will therefore hold, whenever it is $\mu+\nu>\lambda$. Finally, all values, which result from the last integral formula

$$\int \frac{z^{n-1}\partial z}{\sqrt[\lambda]{(1-z^\lambda)^{\lambda-\nu}}},$$

can be considered to be known, whence we will indicate them by the letters A, B, C, D etc.. Therefore, having introduced these values in advance let us in order substitute the numbers 2, 3, 4, 5 etc. for the denominator λ and hence expand the following cases, for which it will be helpful to have observed that in general the numbers μ and ν can be interchanged so that it is

$$\int \frac{z^{\mu-1}\partial z}{\sqrt[\lambda]{(1-z^\lambda)^{\lambda-\nu}}} = \int \frac{z^{\nu-1}\partial z}{\sqrt[\lambda]{(1-z^\lambda)^{\lambda-\mu}}}.$$

I. Expansion of the Case in which it is $\lambda = 2$

§8 For this case our principal equation will therefore be

$$\frac{\mu+\nu}{\mu\nu}\int \partial x\sqrt{u^\mu}\cdot\int \partial x\sqrt{u^\nu} = \int \partial x\sqrt{u^{\mu+\nu}}\cdot\int \frac{z^{\mu-1}\partial z}{(\sqrt{1-zz})^{2-\nu}},$$

since here one can assume only 1 for μ and ν, having put $\mu=1$ and $\nu=1$ for the last formulas only one species results

$$\int \frac{\partial z}{\sqrt{1-zz}},$$

whose value, as it is known, is $=\frac{\pi}{2}$, which because of the analogy to the following cases we will denote by the letter A. Therefore, hence, because it is $\mu+\nu=2$, it will be

$$\int \partial x\sqrt{uu} = \int u\partial x = 1,$$

but the principal equation will obtain this form

$$2\int \partial x\sqrt{u}\cdot\int \partial x\sqrt{u} = \frac{\pi}{2} = A,$$

whence it is

$$\int \partial x\sqrt{u} = \sqrt{\frac{A}{2}} = \frac{1}{2}\sqrt{\pi}.$$

§9 Therefore, since we saw that it is $\int u^{\frac{1}{2}}\partial x = \frac{1}{2}\sqrt{\pi}$, if we continuously augment the exponent of u by 1, by means of the reduction shown above

$$\int u^n \partial x = n \int u^{n-1}\partial x$$

we will obtain the following values

$$\begin{aligned}
\int u^{\frac{3}{2}}\partial x &= \frac{1\cdot 3}{2\cdot 2}\sqrt{\pi}, \\
\int u^{\frac{5}{2}}\partial x &= \frac{1\cdot 3\cdot 5}{2\cdot 2\cdot 2}\sqrt{\pi}, \\
\int u^{\frac{7}{2}}\partial x &= \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 2\cdot 2\cdot 2}\sqrt{\pi}
\end{aligned}$$

and so forth. Further, by going backwards using of the other reduction

$$\int u^n \partial x = \frac{1}{n+1}\int u^{n+1}\partial x$$

we will find

$$\int u^{-\frac{1}{2}}\partial x = \sqrt{\pi}$$

and hence further

$$\begin{aligned}
\int u^{-\frac{3}{2}}\partial x &= -2\sqrt{\pi}, \\
\int u^{-\frac{5}{2}}\partial x &= +\frac{2\cdot 2}{3}\sqrt{\pi}, \\
\int u^{-\frac{7}{2}}\partial x &= -\frac{2\cdot 2\cdot 2}{3\cdot 5}\sqrt{\pi}, \\
\int u^{-\frac{9}{2}}\partial x &= -\frac{2\cdot 2\cdot 2\cdot 2}{3\cdot 5\cdot 7}\sqrt{\pi},
\end{aligned}$$

and so we find the values of our formula for all fractions, whose denominator is $= 2$.

Expansion of the case in which it is $\lambda = 3$

§10 Since here the letters μ and ν can obtain two values, 1 and 2, of course, the last integral formula gives us four values, which we want to denote in the following way

$$\int \frac{\partial z}{\sqrt[3]{1-z^3}} = A, \qquad \int \frac{z\partial z}{\sqrt[3]{1-z^3}} = B,$$
$$\int \frac{\partial z}{\sqrt[3]{(1-z^3)^2}} = A', \qquad \int \frac{z\partial z}{\sqrt[3]{(1-z^3)^2}} = B'.$$

In the first and fourth of these formulas it is $\mu + \nu = \lambda = 3$, whence by the quadrature of the circle we will have

$$A = \frac{\pi}{3\sin\frac{1}{3}\pi} = \frac{2\pi}{3\sqrt{3}} \quad \text{and} \quad B' = \frac{\pi}{3\sin\frac{2}{3}\pi} = \frac{2\pi}{3\sqrt{3}},$$

whence it is plain that it is $B' = A$, which even follows from the fact that the letters μ and ν are interchangeable. Furthermore, note that in the case $\mu + \nu = 3$ it will be

$$\int \partial x \sqrt[3]{u^{\mu+\nu}} = \int u\partial x = 1,$$

but in the case $\mu + \nu = 4$ on the other hand it will be

$$\int \partial x \sqrt[3]{u^4} = \int u^{\frac{4}{3}}\partial x = \frac{4}{3}\int \partial x \sqrt[3]{u}.$$

§11 Having mentioned these things in advance let us expand all cases of our principal equation in the following way:

I. If $\mu = 1$ and $\nu = 2$, it will be $\frac{3}{2}\int \partial x\sqrt[3]{u} \cdot \int \partial x \sqrt[3]{u^2} = A.$

II. If $\mu = 2$ and $\nu = 2$, it will be $\frac{4}{4}\int \partial x\sqrt[3]{u^2} \cdot \int \partial x \sqrt[3]{u^2} = \frac{4}{3}B\int \partial x\sqrt[3]{u}.$

III. If $\mu = 1$ and $\nu = 1$, it will be $2\int \partial x\sqrt[3]{u} \cdot \int \partial x \sqrt[3]{u} = A'\int \partial x\sqrt[3]{u^2}.$

IV. If $\mu = 2$ and $\nu = 1$, it will be $\frac{3}{2}\int \partial x\sqrt[3]{u^2} \cdot \int \partial x \sqrt[3]{u} = B'.$

And so we obtained four equations for the determination of the two unknown values, namely $\int \partial \sqrt[3]{u}$ and $\partial x\sqrt[3]{u^2}$, which can therefore be defined in multiple ways, since only two equations suffice for this.

§12 But to clarify calculation, for the sake of brevity let us set

$$\int \partial x \sqrt[3]{u} = p \quad \text{and} \quad \int \partial x \sqrt[3]{u^2} = q$$

and at first let us combine the equations I and II, which will be

$$\frac{3}{2} pq = A \quad \text{and} \quad qq = \frac{4}{3} Bp;$$

the second of them gives $p = \frac{3qq}{4B}$; this value substituted in the first gives $\frac{9q^3}{8B} = A$, whence one finds

$$q = 2\sqrt[3]{\frac{AB}{9}},$$

from which it is further concluded

$$p = \frac{3}{B}\sqrt[3]{\frac{A^2B^2}{81}}$$

or also

$$p = \sqrt[3]{\frac{AA}{3B}},$$

and so having substituted the values for p and q again we now obtain these equations

$$\int \partial \sqrt[3]{u} = \sqrt[3]{\frac{AA}{3B}} \quad \text{and} \quad 2pp = A'q.$$

From the second it is $q = \frac{2pp}{A'}$, which value substituted in the first gives $\frac{3p^3}{A'} = A$, whence one finds

$$p = \sqrt[3]{\frac{AA'}{3}}$$

and hence

$$q = \frac{2}{A'}\sqrt[3]{\frac{A^2A'A'}{9}} = 2\sqrt[3]{\frac{A^2}{9A'}},$$

and so this combination leads us to these values

$$\int \partial x \sqrt[3]{u} = \sqrt[3]{\frac{AA'}{3}} \quad \text{and} \quad \int \partial \sqrt[3]{u^2} = 2\sqrt[3]{\frac{A^2}{9A'}}.$$

§14 Now let us also combine the first equation with the fourth and we will have $\frac{3}{2}pq = A$ and $\frac{3}{2}pq = B'$, whence only $B' = A$ follows, as we found before. Therefore, let us combine the second with the third and we will have

$$qq = \frac{4}{3}Bp \quad \text{and} \quad 2pp = A'q;$$

from the second of them it is $q = \frac{2pp}{A'}$, which value substituted in the first gives $\frac{4p^3}{A'A'} = \frac{4}{3}B$, whence it is found

$$p = \sqrt[3]{\frac{A'A'B}{3}},$$

whence it is

$$q = 2\sqrt[3]{\frac{A'BB}{9}},$$

and so this combination gives us these values

$$\int \partial x \sqrt[3]{u} = \sqrt[3]{\frac{A'A'B}{3}} \quad \text{and} \quad \int \partial x \sqrt[3]{u^2} = 2\sqrt[3]{\frac{A'BB}{9}}.$$

§15 Since the fourth equation completely agrees with the first, it would be superfluous to combine the second or the third with the fourth, since we already combined them with the first. And so we in total obtained three values for the for the letters p and q, which we want to list up here all together

$$\int \partial x \sqrt[3]{u} = \sqrt[3]{\frac{AA}{3B}} = \sqrt[3]{\frac{A'A'}{3}} = \sqrt[3]{\frac{A'A'B}{3}}$$

and

$$\int \partial x \sqrt[3]{u^2} = 2\sqrt[3]{\frac{AB}{9}} = 2\sqrt[3]{\frac{A^2}{9A'}} = 2\sqrt[3]{\frac{A'BB}{9}}.$$

Therefore, hence having taken the cubes we obtain the equations

$$\frac{AA}{B} = AA' = A'A'B \quad \text{and} \quad AB = \frac{AA}{A'} = A'BB.$$

§16 But having related all these different values all these equalities are reduced to this single one

$$A = A'B'$$

Therefore, having substituted the integral formulas themselves we obtain this most remarkable relation

$$\int \frac{\partial z}{\sqrt[3]{1-z^3}} = \int \frac{\partial z}{\sqrt[3]{(1-z^3)^2}} \cdot \int \frac{z\partial z}{\sqrt[3]{1-z^3}},$$

and since A is defined by the quadrature of the circle, the value of this product will result as

$$\int \frac{\partial z}{\sqrt[3]{(1-z^3)^2}} \cdot \int \frac{z\partial z}{\sqrt[3]{1-z^3}} = \frac{2\pi}{3\sqrt{3}},$$

whence, if the one of these two formulas would be known, at the same time the value of the other would be known; for, this way from two values A and B the two remaining ones A' and B' are determined in such a way that it is

$$A' = \frac{A}{B} \quad \text{and} \quad B' = A.$$

Finally, it will also be worth one's while to have noted this relation

$$\int \partial x \sqrt[3]{u} \cdot \int \partial x \sqrt[3]{u^2} = \frac{2}{3}A = \frac{4\pi}{9\sqrt{3}}.$$

Expansion of the Case in which it is $\lambda = 4$

§17 Here, for the sake of brevity let us first put

$$\int \partial x \sqrt[4]{u} = p, \quad \int \partial x \sqrt[4]{u^2} = q \quad \text{and} \quad \int \partial x \sqrt[4]{u^3} = r;$$

furthermore, let us denote the integral formula

$$\int \frac{z^{\mu-1}\partial z}{\sqrt[4]{(1-z^4)^{4-\nu}}}$$

by the character (μ, ν), since we already saw that the letters μ and ν can be interchanged. Additionally, represent the principal equation this way

$$\int \partial x \sqrt[4]{u^{\mu}} \cdot \int \partial x \sqrt[4]{u^{\nu}} = \frac{\mu\nu}{\mu+\nu} \int \partial x \sqrt[4]{u^{\mu+\nu}} \cdot (\mu, \nu),$$

where it should be noted, if $\mu + \nu = \lambda = 4$, that it will be

$$\int \partial x \sqrt[4]{u^4} = 1;$$

but if $\mu + \nu = \lambda + \alpha = 4 + \alpha$, it will be

$$\int \partial x \sqrt[4]{u^{4+\alpha}} = \left(1 + \frac{\alpha}{4}\right) \int u^{\frac{\alpha}{4}} \partial x = \frac{\mu+\nu}{4} \int \partial x \sqrt[4]{u^{\alpha}}.$$

§18 Now let us successively attribute all values smaller than 4 to the letters μ and ν and the principal equation will give us the following equations:

1°. If $\begin{pmatrix} \mu = 1 \\ \nu = 1 \end{pmatrix}$, it will be $pp = \frac{1}{2}q(1,1)$, whence it is $\frac{pp}{q} = \frac{1}{2}(1,1) = A$.

2°. If $\begin{pmatrix} \mu = 1 \\ \nu = 2 \end{pmatrix}$, it will be $pq = \frac{2}{3}r(1,2)$, whence it is $\frac{pq}{r} = \frac{2}{3}(1,2) = B$.

3°. If $\begin{pmatrix} \mu = 1 \\ \nu = 3 \end{pmatrix}$, it will be $pr = \frac{3}{4}(1,3) = C$.

4°. If $\begin{pmatrix} \mu = 1 \\ \nu = 4 \end{pmatrix}$, it will be $qq = (2,2) = D$.

5°. If $\begin{pmatrix} \mu = 2 \\ \nu = 3 \end{pmatrix}$, it will be $qr = \frac{6}{5} \cdot \frac{5}{4}p(2,3)$, whence it is $\frac{qr}{p} = \frac{3}{2}(2,3) = E$.

6°. If $\begin{pmatrix} \mu = 3 \\ \nu = 3 \end{pmatrix}$, it will be $rr = \frac{9}{6} \cdot \frac{6}{4}q(3,3)$, whence it is $\frac{rr}{q} = \frac{9}{4}(3,3) = F$.

§19 Therefore, we hence obtain six equations, from which our three unknowns p, q and r must be defined, which can therefore be done in several ways, since three equations suffice. Therefore, let us chose those, which solve the problem most conveniently; and first the fourth immediately gives us

$$q = \sqrt{D},$$

whence from the first we find $pp = A\sqrt{D}$ and hence

$$p = \sqrt{A}\sqrt{D} = \sqrt[4]{AAD},$$

finally we conclude $rr = F\sqrt{D}$ from the sixth equation and hence

$$r = \sqrt[4]{FFD}$$

and so we determined all three transcendental formulas in such a way that it is

$$\begin{aligned}
&1^\circ. \quad p = \int \partial x \sqrt[4]{u} \;\; = \sqrt[4]{AAD} \;\; = \sqrt[4]{\frac{1}{4}(1,1)^2(2,2)}\\
&2^\circ. \quad q = \int \partial x \sqrt[4]{u^2} = \int \partial x \sqrt{u} = \sqrt{D} = \sqrt{(2,2)},\\
&3^\circ. \quad r \; = \int \partial x \sqrt[4]{u^3} = \sqrt[4]{DFF} \;\; = \frac{3}{2}\sqrt[4]{(2,2)(3,3)^2}.
\end{aligned}$$

§20 Here it will now be helpful to have noted that the value of the formula (μ, ν) in the case, in which it is $\mu + \nu = \lambda$, can in general be expressed by the quadrature of the circle, since in this case it is

$$(\mu, \nu) = \frac{\pi}{\lambda \frac{\mu\pi}{\lambda}}.$$

Therefore, in our case, in which it is $\lambda = 4$, it will be

$$(2,2) = \frac{\pi}{4 \sin \frac{\pi}{2}} = \frac{\pi}{4}$$

further, it will also be

$$(1,3) = \frac{\pi}{4 \sin \frac{1}{4}\pi} = \frac{\pi}{2\sqrt{2}}.$$

Therefore, it is hence plain that it will be

$$D = \frac{\pi}{4} \quad \text{and} \quad C = \frac{3}{4} \cdot \frac{\pi}{2\sqrt{2}} = \frac{3\pi}{8\sqrt{2}},$$

so that these two letters C and D depend only on the quadrature of the circle.

§21 Since we found these three determinations, namely

$$p = \sqrt[4]{AAD}, \quad q = \sqrt{D} \quad \text{and} \quad r = \sqrt[4]{DFF},$$

from the equations 1,4 and 6, if we substitute the same values in the remaining equations, we will find extraordinary relations among our small letters. For, so the second equation $pq = Br$ will give $AAD^3 = B^4DFF$, which is reduced to this one

$$AD = BBF;$$

the third equation $pr = C$ on the other hand will give

$$ADF = CC;$$

finally, the fifth equation $qr = Ep$ will yield $D^3FF = A^2DE^4$, whence it is

$$DF = AEE.$$

Therefore, this way we are led to the three following relations

$$1^\circ. \quad AD = BBF, \quad 2^\circ. \quad ADF = CC \quad \text{and} \quad 3^\circ. \quad DF = AEE,$$

the first of which multiplied by the second will give $AD = BC$, but the second on the other hand multiplied by the third produces $DF = CE$. Therefore, because it is $AD = BC$, from the first it is concluded that it will also be $C = BF$, so that the found equations are reduced to these three

$$1^\circ. \quad C = AE, \quad 2^\circ. \quad C = BF, \quad 3^\circ. \quad AD = BC,$$

which are reduced to these three most simple ones

$$1^\circ. \quad C = AE, \quad 2^\circ. \quad C = BF, \quad 3^\circ. \quad D = BE.$$

§22 Therefore, if in these last equations we introduce the integral formulas denoted by our characters instead of the letters, the following relations will result:

$$\begin{aligned} 1^\circ. &\quad (1,3) = (1,1)(2,3), \\ 2^\circ. &\quad (1,3) = 2(1,2)(3,3), \end{aligned}$$

and

$$3^\circ. \quad (2,2) = (1,2)(2,3).$$

Therefore, hence by means of the integral formulas we will have these three most memorable relations:

$$1^\circ. \quad \frac{\pi}{2\sqrt{2}} = \int \frac{\partial z}{\sqrt[4]{(1-z^4)^3}} \cdot \int \frac{z\partial z}{\sqrt[4]{1-z^4}} = \int \frac{\partial z}{\sqrt[4]{(1-z^4)^3}} \cdot \int \frac{zz\partial z}{\sqrt{1-z^4}},$$

$$2^\circ. \quad \frac{\pi}{4\sqrt{2}} = \int \frac{\partial z}{\sqrt{1-z^4}} \cdot \int \frac{zz\partial z}{\sqrt[4]{1-z^4}}$$

and

$$3^\circ. \quad \frac{\pi}{4} = \int \frac{\partial z}{\sqrt{1-z^4}} \cdot \int \frac{zz\partial z}{\sqrt{1-z^4}},$$

the last of which I published already a long time ago[3].

§23 Therefore, because from the six integral formulas, which occur here, two, namely C and D, depend on the quadrature of the circle, if only one of the remaining ones becomes known, the values of all the others can be assigned from this. For, if except for the characters $(1,3)$ and $(2,2)$ we additionally consider this one $(1,2)$ to be known, the remaining three will be determined by these three in the following way:

$$(3,3) = \frac{(1,3)}{2(1,2)}, \quad (2,3) = \frac{(2,2)}{(1,2)}, \quad (1,1) = \frac{(1,2)(1,3)}{(2,2)}.$$

EXPANSION OF THE CASE IN WHICH IT IS $\lambda = 5$

§24 Here, let us call the transcendental formulas in question

$$\int u^{\frac{1}{5}}\partial x = p, \quad \int u^{\frac{2}{5}}\partial x = q, \quad \int u^{\frac{3}{5}}\partial x = r, \quad \int u^{\frac{4}{5}}\partial x = s.$$

Now, let the character (μ, ν) denote this integral formula

$$\int \frac{z^{\mu-1}\partial z}{\sqrt[5]{(1-z^5)^{5-\nu}}};$$

having constituted all this from the principle equation we will obtain the following equations:

[3]Euler refers to his paper "De productis ex infinitis factoribus ortis". This is paper E122 in the Eneström-Index.

$$
\begin{array}{lllllll}
1^\circ. & \text{If} & \begin{pmatrix}\mu=1\\ \nu=1\end{pmatrix}, & \text{it will be} & pp=\frac{1}{2}q(1,1), & \text{whence it is} & \frac{pp}{q}=\frac{1}{2}(1,1)\ =A.\\
2^\circ. & \text{If} & \begin{pmatrix}\mu=1\\ \nu=2\end{pmatrix}, & \text{it will be} & pq=\frac{2}{3}r(1,2), & \text{therefore} & \frac{pq}{r}=\frac{2}{3}(1,2)\ =B.\\
3^\circ. & \text{If} & \begin{pmatrix}\mu=1\\ \nu=3\end{pmatrix}, & \text{it will be} & pr=\frac{3}{4}s(1,3) & \text{therefore} & \frac{pr}{s}=\frac{3}{4}(1,3)\ =C\\
4^\circ. & \text{If} & \begin{pmatrix}\mu=1\\ \nu=4\end{pmatrix}, & \text{it will be} & ps=\frac{4}{5}(1,4)\ =\ D. & & \\
5^\circ. & \text{If} & \begin{pmatrix}\mu=2\\ \nu=2\end{pmatrix}, & \text{it will be} & qq=s(2,2) & \text{therefore} & \frac{qq}{s}=(2,2)\ =E.\\
6^\circ. & \text{If} & \begin{pmatrix}\mu=2\\ \nu=3\end{pmatrix}, & \text{it will be} & qr=\frac{6}{5}(2,3)\ =\ F. & & \\
7^\circ. & \text{If} & \begin{pmatrix}\mu=2\\ \nu=4\end{pmatrix}, & \text{it will be} & qs=\frac{8}{6}\cdot\frac{6}{5}p(2,4) & \text{therefore} & \frac{qs}{p}=\frac{8}{5}(2,4)\ =G.\\
8^\circ. & \text{If} & \begin{pmatrix}\mu=3\\ \nu=3\end{pmatrix}, & \text{it will be} & rr=\frac{9}{6}\cdot\frac{6}{5}p(3,3) & \text{therefore} & \frac{rr}{p}=\frac{9}{5}(3,3)\ =H.\\
9^\circ. & \text{If} & \begin{pmatrix}\mu=3\\ \nu=4\end{pmatrix}, & \text{it will be} & rs=\frac{12}{7}\cdot\frac{7}{5}q(3,4) & \text{therefore} & \frac{rs}{p}=\frac{12}{5}(3,4)=I.\\
10^\circ. & \text{If} & \begin{pmatrix}\mu=4\\ \nu=4\end{pmatrix}, & \text{it will be} & ss=\frac{16}{8}\cdot\frac{8}{5}r(4,4) & \text{therefore} & \frac{ss}{r}=\frac{16}{5}(4,4)=K.
\end{array}
$$

§25 Therefore, since we obtained ten equations, from which four unknown quantities must be determined, let us choose those, by which the problem will be solved most easily. But the fourth equation immediately gives

$$s=\frac{D}{p};$$

but from the sixth it is

$$r=\frac{F}{q},$$

so that it remains to find the two letters p and q. Further, from the first on the other hand we deduce

$$q = \frac{pp}{A},$$

so that it is

$$r = \frac{AF}{pp}.$$

Therefore, now from the second equation it will be

$$\frac{p^5}{AAF} = B,$$

whence it is

$$p = \sqrt[5]{AABF};$$

having found this value it is concluded that it will be

$$q = \sqrt[5]{\frac{BBFF}{A}}, \quad r = \sqrt[5]{\frac{AF^3}{BB}};$$

finally, it will be

$$s = \frac{D}{\sqrt[5]{AABF}}.$$

And so all four unknown quantities can be expressed by means of ordinary quadratures. If we now substitute these values in the remaining equations, the following equations will result:

$$
\begin{aligned}
&1^\circ. \quad CD = AF,\\
&2^\circ. \quad BF = ED,\\
&3^\circ. \quad D = AG,\\
&4^\circ. \quad F = BH,\\
&5^\circ. \quad D = BI,\\
&6^\circ. \quad DD = AFK,
\end{aligned}
$$

whence because of $D = AG$ one finds

$$DG = FK.$$

§26 Therefore, lo and behold, six new determinations resulted, relating our ten letters to each other, so that from four assumed to be known the remaining six can be defined; but it will be especially convenient to assume the two D and F as known, which are expressed by the quadrature of the circle, of course, since it is

$$D = \frac{4}{5}(1,4) = \frac{4}{5} \cdot \frac{\pi}{5 \sin \frac{1}{5}\pi} \quad \text{and} \quad F = \frac{6}{5}(2,3) = \frac{6}{5} \cdot \frac{\pi}{5 \sin \frac{2}{5}\pi}.$$

Therefore, as long as two of the remaining ones are also considered to be known, it will be possible to define all others by means of them. But on the other hand those six relations compared to each other correctly yield three formulas equal so to D as to F, which are

$$D = AG = BI = CK \quad \text{and} \quad F = BH = CG = EI.$$

Therefore, hence, if except for D and F also the letters A and B are assumed to be known, the remaining letters are determined from them as follows:

$$C = \frac{AF}{D}, \quad E = \frac{BF}{D}, \quad G = \frac{D}{A}, \quad H = \frac{F}{B}, \quad I = \frac{D}{B} \quad \text{and} \quad K = \frac{DD}{AF}.$$

§27 Now, let us substitute the characters of the integral formulas for these letters and the following six relations will be obtained:

$$\begin{aligned}
&1^\circ \quad (1,4) = (1,1)(2,4),\\
&2^\circ \quad (1,4) = 2(1,2)(3,4),\\
&3^\circ \quad (1,4) = 3(1,3)(4,4),\\
&4^\circ \quad (2,3) = (1,2)(3,3),\\
&5^\circ \quad (2,3) = (1,3)(2,4),\\
&6^\circ \quad (2,3) = 2(2,2)(3,4);
\end{aligned}$$

hence many extraordinary theorems could be formulated.

§28 Since both letters D and F or rather the characters $(1,4)$ and $(2,3)$ involve the circumference of the circle, their ratio $\frac{(1,4)}{(2,3)}$ can be exhibited algebraically, whose value is

$$= \frac{\sin\frac{2}{5}\pi}{\sin\frac{1}{5}\pi} = 2\cos\frac{1}{5}\pi.$$

Hence also the following ratios of two integral formulas are derived

$$2\cos\frac{1}{5}\pi = \frac{(1,1)}{(1,3)} = 2\frac{(3,4)}{(3,3)} = \frac{(1,2)}{(2,2)} = 3\frac{(4,4)}{(2,4)},$$

whence again extraordinary theorems could be formulated, if the present undertaking would demand it. But I will delay the more complete explanation of this subject to another occasion.

§29 In like manner, as we expanded the case $\lambda = 5$ here, it would be possible to treat the following cases, in which larger values are attributed to the letter λ. But since the number of equations always increases according to the triagonal numbers, it would be superfluous to invest the work here, since all analytic operations, on which these solutions are based, are already explained sufficiently diligently.